

**q -IDENTITIES AND AFFINIZED PROJECTIVE VARIETIES
I. QUADRATIC MONOMIAL IDEALS**

PETER BOUWKNEGT

ABSTRACT. We define the concept of an affinized projective variety and show how one can, in principle, obtain q -identities by different ways of computing the Hilbert series of such a variety. We carry out this program for projective varieties associated to quadratic monomial ideals. The resulting identities have applications in describing systems of quasi-particles containing null-states and can be interpreted as alternating sums of quasi-particle Fock space characters.

1. INTRODUCTION

The topic of q -identities, such as the Rogers-Ramanujan identities, has attracted a lot of attention throughout the last century or so. Initially, mostly in connection to the theory of partitions (see, e.g., [An]), later in connection with the representation theory of infinite dimensional Lie algebras (see, e.g., [Ka]). Recently, there has been a surge of new research in this area instigated by the discovery by the ‘Stony Brook group’ of certain ‘fermionic-type’ (or quasi-particle type) formulas for the (chiral) partition functions of two-dimensional conformal field theories (cf., in particular, the reviews [DKKMM,KMM] and references therein).

There are many techniques for finding and/or proving q -identities such as classical techniques by combinatorics, generating series, recursion relations as well as more modern ones based on Bailey’s transform, crystal bases, spinon bases and path representations. The aim of this paper is to explain yet another technique based on the relation between certain q -identities and the geometry of so-called affinized projective varieties, in particular through the computation of the Hilbert series of the (homogeneous) coordinate ring of such varieties. A relation between q -identities and the geometry of infinite dimensional varieties has also been put forward in [FS].

One of the simplest examples of the type of identity we have in mind is

$$(1.1) \quad \frac{q^{M_1 M_2}}{(q)_{M_1} (q)_{M_2}} = \sum_{m \geq 0} (-1)^m \frac{q^{\frac{1}{2} m(m-1)}}{(q)_m (q)_{M_1-m} (q)_{M_2-m}},$$

where

$$(1.2) \quad (q)_N = \prod_{k=1}^N (1 - q^k).$$

The identity (1.1) first arose in the spinon description of $\widehat{\mathfrak{sl}}_3$ modules [BS1,BS2]. The alternating sum on the right hand side of (1.1) indicates the presence of null-states in the spinon Fock space which are removed by inclusion-exclusion (sieving). In this paper we will explain the geometric origin of the identity (1.1) and an algorithm for constructing a host of identities of similar (alternating) type which have similar interpretations as alternating sums over quasi-particle Fock space characters.

To get some insight in the geometric origin of this relation, multiply both sides by $y_1^{M_1} y_2^{M_2}$ and sum over $M_1, M_2 \geq 0$. Then consider the $\mathcal{O}(q^0)$ -term on each side of the equation. On the left hand side we have contributions from those $M_1, M_2 \geq 0$ such that $M_1 M_2 = 0$, i.e., either $M_1 \geq 0, M_2 = 0$, or $M_1 = 0, M_2 \geq 1$, while on the right hand side only the $m = 0, 1$ terms contribute. At $\mathcal{O}(q^0)$ we thus find the (obvious) identity

$$(1.3) \quad \frac{1}{1-y_1} + \frac{y_2}{1-y_2} = \frac{1}{(1-y_1)(1-y_2)} - \frac{y_1 y_2}{(1-y_1)(1-y_2)}.$$

Alternatively, (1.3) arises from two different ways of computing the Hilbert series of the projective variety V consisting of 2 points in \mathbb{P}^1 . The left hand side of (1.3) is computed by constructing an explicit basis for the homogeneous coordinate ring $\mathbb{C}[x_1, x_2]/\langle x_1 x_2 \rangle$ of V , while the right hand side arises from a free resolution of this coordinate ring. In this paper we will argue that (1.1) arises, in a similar way, from an appropriately defined affinization of the variety V .

The paper is organized as follows. In section 2 we give a basic review of some elementary concepts involving (projective) varieties, Hilbert series and resolutions of monomial ideals in polynomial rings. This section serves mainly to establish notations and to make the paper accessible to an audience without expertise in algebraic geometry. In section 3 we introduce the concept of an affinized projective variety and its associated Hilbert series and illustrate their use in the example which leads to (1.1). In section 4 we explain an algorithm which leads to a q -identity for any projective variety associated to a quadratic monomial ideal. In section 5 we illustrate the algorithm by explicitly going through some examples. In section 6 we conclude with some remarks regarding the existence and nature of q -identities associated to more general ideals.

In a sequel to this paper we will apply our ideas to the q -identities associated to flag varieties and their connection to the representation theory of affine Lie algebras and modified Hall-Littlewood polynomials [BH].

2. PROJECTIVE VARIETIES AND HILBERT SERIES

2.1. Varieties versus ideals.

We begin by summarizing some elementary facts from algebraic geometry (see, e.g., [Ha,Ei,CLO1,CLO2]). Throughout this paper we will work over the field \mathbb{C} of complex numbers.

An affine variety $V \subset \mathbb{C}^n$ is the zero locus of a set of polynomials, f_1, \dots, f_t , in the coordinate ring, $\mathbb{C}[x_1, \dots, x_n]$, of \mathbb{C}^n , i.e.,

$$(2.1) \quad V = \{(x_1, \dots, x_n) \in \mathbb{C}^n : f_i(x_1, \dots, x_n) = 0 \text{ for all } 1 \leq i \leq t\}.$$

There is a close correspondence between affine varieties $V \subset \mathbb{C}^n$ and ideals in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$. Namely, for $V \subset \mathbb{C}^n$, we can define an ideal

$$(2.2) \quad \mathbf{I}(V) = \{f \in \mathbb{C}[x_1, \dots, x_n] : f(x_1, \dots, x_n) = 0 \text{ for all } (x_1, \dots, x_n) \in V\}$$

while, conversely, for an ideal $I \in \mathbb{C}[x_1, \dots, x_n]$, we can define the set

$$(2.3) \quad \mathbf{V}(I) = \{(x_1, \dots, x_n) \in \mathbb{C}^n : f(x_1, \dots, x_n) = 0 \text{ for all } f \in I\}.$$

That $\mathbf{V}(I)$ is actually an affine variety is assured by Hilbert's basis theorem which states that every ideal $I \in \mathbb{C}[x_1, \dots, x_n]$ has a finite generating set. Clearly, for any affine variety $V \subset \mathbb{C}^n$ we have $\mathbf{V}(\mathbf{I}(V)) = V$. The converse is not true however. The composition $\mathbf{I} \circ \mathbf{V}$ is neither injective nor surjective. Precisely which I appear in the image of $\mathbf{I} \circ \mathbf{V}$ is settled by Hilbert's Nullstellensatz, which states

$$(2.4) \quad \mathbf{I}(\mathbf{V}(I)) = \sqrt{I} \equiv \{f \in \mathbb{C}[x_1, \dots, x_n] : \exists r > 0, f^r \in I\}.$$

Thus, there exists a 1–1 correspondence between affine varieties $V \subset \mathbb{C}^n$ and radical ideals $I \in \mathbb{C}[x_1, \dots, x_n]$, i.e., ideals for which $\sqrt{I} = I$.

Let $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$ denote the (complex) projective space. We will use homogeneous coordinates $[x_1, \dots, x_n]$ for \mathbb{P}^{n-1} . Then, a projective variety $V \subset \mathbb{P}^{n-1}$ is the zero locus of a set of homogeneous polynomials f_1, \dots, f_t in the homogeneous coordinate ring $\mathbb{C}[x_1, \dots, x_n]$ of \mathbb{P}^{n-1} . In analogy with the above, we now have a correspondence between projective varieties and homogeneous ideals $I \subset \mathbb{C}[x_1, \dots, x_n]$ (see, e.g., [CLO1, Chapter 8] for more details).

2.2. Hilbert series.

Consider a homogeneous ideal $I \subset \mathbf{S} = \mathbb{C}[x_1, \dots, x_n]$. Let $\mathbf{S}(V) = \mathbf{S}/I$ denote the homogeneous coordinate ring of the associated projective variety $\mathbf{V}(I)$ and let $\mathbf{S}(V)_M$ denote the vector space of homogeneous polynomials of degree M in $\mathbf{S}(V)$. The function

$$(2.5) \quad h_V(M) = \dim \mathbf{S}(V)_M,$$

is called the (projective) Hilbert function of V . One can prove that there exists a polynomial $p_V(M)$ such that for M sufficiently large we have $p_V(M) = h_V(M)$. The polynomial $p_V(M)$ contains important information about the variety V , e.g., the degree of $p_V(M)$ is the dimension of V . In this paper we will also use the Hilbert series $h_V(y)$ of V , i.e., generating series for $h_V(M)$

$$(2.6) \quad h_V(y) = \sum_{M \geq 0} h_V(M) y^M.$$

By slight abuse of notation we will also denote by $h_V(y)$ the Hilbert series of any \mathbf{S} -module V .

For any \mathbf{S} -module M , let $M(a)$ denote the same module with the degree shifted by a . Clearly, for $M \geq a$,

$$(2.7) \quad \dim \mathbf{S}(-a)_M = \dim \mathbf{S}_{M-a} = \binom{M-a+n}{n},$$

so that

$$(2.8) \quad h_{\mathbf{S}(-a)}(y) = \sum \dim \mathbf{S}(-a)_M y^M = \frac{y^a}{(1-y)^n}.$$

There are (at least) two methods to explicitly compute the Hilbert series of a projective variety $\mathbf{V}(I)$. The first is by constructing an explicit basis for the \mathbf{S} -module $\mathbf{S}(V) = \mathbf{S}/I$ (see, e.g., [CLO1, Chapter 9] for a recipe in the case of monomial ideals). The second is by means of a free resolution of the \mathbf{S} -module $\mathbf{S}(V)$. The existence of a (finite length) free resolution of $\mathbf{S}(V)$, i.e., an exact sequence

$$(2.9) \quad 0 \rightarrow F^{(\nu)} \rightarrow \dots \xrightarrow{d_3} F^{(2)} \xrightarrow{d_2} F^{(1)} \xrightarrow{d_1} F^{(0)} \cong \mathbf{S} \rightarrow \mathbf{S}(V) \rightarrow 0,$$

where each $F^{(i)} = \bigoplus_j \mathbf{S}(-a_j^{(i)})$ for some set of positive integers $a_j^{(i)}$, is guaranteed by Hilbert's syzygy theorem. Applying the Euler-Poincaré principle to the resolution (2.9) yields

$$(2.10) \quad h_V(y) = \sum_{i \geq 0} (-1)^i h_{F^{(i)}}(y) = \sum_{i,j} (-1)^i \frac{y^{a_j^{(i)}}}{(1-y)^n},$$

where we have used (2.8).

In case the ideal I is homogenous in various subsets of coordinates separately, the quotient module \mathbf{S}/I carries a multi-degree $M = (M_1, \dots, M_s)$. The above constructions then have an obvious multi-degree generalization.

As an example consider the variety V defined by the ideal $I = \langle x_1 x_2 \rangle \subset \mathbb{C}[x_1, x_2]$. This variety consists of two points in \mathbb{P}^1 . The coordinate ring $\mathbf{S}(V) = \mathbb{C}[x_1, x_2]/\langle x_1 x_2 \rangle$ carries a bi-degree $\deg(x_1) = (1, 0)$, $\deg(x_2) = (0, 1)$. Obviously, $\mathbf{S}(V)$ has a basis $\{x_1^m, m \geq 0\} \cup \{x_2^m, m > 0\}$ so that

$$(2.11) \quad h_V(y) = \frac{1}{1-y_1} + \frac{y_2}{1-y_2}.$$

On the other hand, the resolution of $\mathbf{S}(V)$ looks like

$$(2.12) \quad 0 \rightarrow \mathbf{S}(-1, -1) \xrightarrow{d_1} \mathbf{S} \rightarrow \mathbf{S}(V) \rightarrow 0,$$

where the map $d_1 : \mathbf{S}(-1, -1) \rightarrow \mathbf{S}$ is defined as $P \mapsto (x_1 x_2)P$. So, the multi-degree generalization of (2.10) leads to

$$(2.13) \quad h_V(y) = \frac{1}{(1-y_1)(1-y_2)} - \frac{y_1 y_2}{(1-y_1)(1-y_2)}.$$

The equality of (2.11) and (2.13) is obvious (cf. (1.3)).

2.3. Taylor's resolution of a monomial ideal.

In this section we recall a resolution of monomial ideals due to Taylor (see [Ei, Exercise 17.11]). Suppose we have an ideal $\langle f_1, \dots, f_t \rangle \subset \mathbb{C}[x_1, \dots, x_n] \equiv \mathbf{S}$ generated by monomials f_i , $i = 1, \dots, t$. Let \mathcal{I}_s be the set of (ordered) subsets of $\{1, \dots, t\}$ of length s , i.e., $I \in \mathcal{I}_s$ is an s -tuple $\{i_1, \dots, i_s\} \subset \{1, 2, \dots, t\}$ with $i_1 < \dots < i_s$. We will also denote $\mathcal{I} = \bigcup_s \mathcal{I}_s$ and $|I| = s$ for $I \in \mathcal{I}_s$. Let $F^{(s)}$ be the free \mathbf{S} -module on basis elements e_I , $I \in \mathcal{I}_s$, and, for $I \in \mathcal{I}_s$, let

$$(2.14) \quad f_I = \text{LCM}\{f_i : i \in I\}$$

where $\text{LCM}\{f_i\}$ denotes the lowest common multiple of the monomials f_i , $i \in I$. Furthermore, for $I = \{i_1, \dots, i_s\} \in \mathcal{I}_s$ and $J \in \mathcal{I}_{s-1}$ we define

$$(2.15) \quad c_{IJ} = \begin{cases} 0 & \text{if } J \not\subset I, \\ (-1)^k f_I / f_J & \text{if } I = J \cup \{i_k\} \text{ for some } k. \end{cases}$$

We have maps $d_s : F^{(s)} \rightarrow F^{(s-1)}$ defined by

$$(2.16) \quad d_s : e_I \mapsto \sum_{J \in \mathcal{I}_{s-1}} c_{IJ} e_J,$$

satisfying $d_{s-1}d_s = 0$. The corresponding complex

$$(2.17) \quad 0 \rightarrow F^{(t)} \xrightarrow{d_t} \dots \xrightarrow{d_2} F^{(1)} \xrightarrow{d_1} F^{(0)} \cong \mathbf{S} \rightarrow \mathbf{S}(V) \rightarrow 0,$$

gives a free resolution of $\mathbf{S}(V)$.

Remark 2.1. If $f_{i_1, \dots, i_s} = f_{i_1} \dots f_{i_s}$ for all $I = \{i_1, \dots, i_s\} \in \mathcal{I}$, the resolution (2.17) is a so-called Koszul resolution and the corresponding variety $\mathbf{V}(I)$ is called a complete intersection (cf. [Ha, Example 13.16]). In general, there may exist subsets $J, J' \in \mathcal{I}$ such that $f_I = f_J f_{J'}$ and $J \cup J' \subset I$. We will refer to these as ‘Koszul parts’ of Taylor’s resolution. They are usually an indication that the resolution (2.17) is not a minimal resolution.

As an example, consider the ideal $I = \langle x_1x_2, x_2x_3, x_3x_4 \rangle \subset \mathbb{C}[x_1, x_2, x_3, x_4]$ (cf. section 5.2). Put $f_1 = x_1x_2$, $f_2 = x_2x_3$ and $f_3 = x_3x_4$. We find

$$(2.18) \quad \begin{aligned} f_{12} &= x_1x_2x_3, & f_{23} &= x_2x_3x_4, & f_{13} &= x_1x_2x_3x_4, \\ f_{123} &= x_1x_2x_3x_4, \end{aligned}$$

so Taylor’s resolutions (2.17) is given by

$$(2.19) \quad \begin{aligned} F^{(1)} &\cong \widehat{\mathbf{S}}e_1 \oplus \widehat{\mathbf{S}}e_2 \oplus \widehat{\mathbf{S}}e_3 \\ &\cong \widehat{\mathbf{S}}(-1, -1, 0, 0) \oplus \widehat{\mathbf{S}}(0, -1, -1, 0) \oplus \widehat{\mathbf{S}}(0, 0, -1, -1), \\ F^{(2)} &\cong \widehat{\mathbf{S}}e_{12} \oplus \widehat{\mathbf{S}}e_{23} \oplus \widehat{\mathbf{S}}e_{13} \\ &\cong \widehat{\mathbf{S}}(-1, -1, -1, 0) \oplus \widehat{\mathbf{S}}(0, -1, -1, -1) \oplus \widehat{\mathbf{S}}(-1, -1, -1, -1), \\ F^{(3)} &\cong \widehat{\mathbf{S}}e_{123} \\ &\cong \widehat{\mathbf{S}}(-1, -1, -1, -1), \end{aligned}$$

and maps d_s given by (2.16). The minimal resolution is however given by removing the spaces $\widehat{\mathbf{S}}(-1, -1, -1, -1)$ from $F^{(2)}$ and $F^{(3)}$, as one can easily see.

3. AFFINIZED PROJECTIVE VARIETIES AND q -IDENTITIES

Consider a projective variety $V \subset \mathbb{P}^{n-1}$, defined by the ideal $\mathbf{I}(V)$ generated by a set of homogeneous elements f_i , $i = 1, \dots, t$. By the affinized projective variety $\widehat{V} \subset \widehat{\mathbb{P}^{n-1}}$ we mean the infinite projective variety defined by the ideal $\mathbf{I}(\widehat{V})$ generated by the relations $f_i[x] = 0$, $i = 1, \dots, t$, in $\widehat{\mathbf{S}} = \mathbb{C}[x_1, \dots, x_n]$.

$\mathbb{C}[[t]] = \mathbb{C}[x_1[m], \dots, x_n[m]]_{m \in \mathbb{Z}_{\geq 0}}$ where we have written $x_i[m] = x_i \otimes t^m$, and where $f_i[m]$ is obtained from f_i by replacing all monomials $x_{i_1} \dots x_{i_r}$ by

$$(3.1) \quad (x_{i_1} \dots x_{i_r})[m] = \sum_{\substack{n_1, \dots, n_r \geq 0 \\ n_1 + \dots + n_r = m}} x_{i_1}[n_1] \dots x_{i_r}[n_r].$$

The coordinate ring $\mathbf{S}(\widehat{V})$ of the affinized projective variety \widehat{V} is graded by the multi-degree defined by

$$(3.2) \quad \deg(x_i[m]) = (\deg(x_i); m),$$

i.e., both by the (multi-) degree inherited from the underlying projective variety V , as well as the ‘energy’ m .

We denote by $\mathbf{S}(\widehat{V})_{(M;N)}$ the vector space of homogeneous polynomials f of multi-degree $(M; N)$ in $\mathbf{S}(\widehat{V})$. By analogy with (2.5), the Hilbert function is defined as $h_{\widehat{V}}(M; N) = \dim \mathbf{S}(\widehat{V})_{(M;N)}$. Note that the introduction of ‘energy’ makes $h_{\widehat{V}}(M; N)$ finite. The Hilbert series of \widehat{V} is defined as

$$(3.3) \quad h_{\widehat{V}}(y; q) = \sum_{M, N} h_{\widehat{V}}(M; N) y^M q^N.$$

We will also be using the partial Hilbert series

$$(3.4) \quad h_{\widehat{V}}(M; q) = \sum_N h_{\widehat{V}}(M; N) q^N.$$

Remark 3.1. Note that $\mathbf{S}(V) \subset \mathbf{S}(\widehat{V})$ through the identification $x_i \sim x_i[0]$. Therefore we have the obvious equality $h_{\widehat{V}}(M; 0) = h_V(M)$ between the Hilbert function of V and the energy $N = 0$ Hilbert function of \widehat{V} , i.e., the $\mathcal{O}(q^0)$ -term in the partial Hilbert series of \widehat{V} .

Remark 3.2. The variables $x_i[m]$ combine into ‘currents’

$$x_i(t) = \sum_{m \geq 0} x_i[m] t^m.$$

In terms of these currents, the ‘energy’ is just the eigenvalue of the derivation $t \frac{d}{dt}$ while the ideal \widehat{I} is generated by the modes of currents $f_i(t)$ which are compositions of the $x_i(t)$. The Hilbert series (3.4) has the interpretation of a $U(1)$ character.

As in the finite dimensional case, there are in principle two different ways of computing the Hilbert series of \widehat{V} . On the one hand, we may be able to construct an explicit basis for the coordinate ring $\mathbf{S}(\widehat{V})$ of \widehat{V} . On the other hand we may compute $h_{\widehat{V}}(y; q)$ by applying the Eurler-Poincaré principle to a free resolution of $\mathbf{S}(\widehat{V})$

$$(3.5) \quad d_3 \cdot \Gamma(2) \cdot d_2 \cdot \Gamma(1) \cdot d_1 \cdot \widehat{\mathbf{S}} \rightarrow \mathbf{S}(\widehat{V}) \rightarrow 0$$

respecting the grading by the multi-degree (3.2). Of course, in the affinized case, the resolution (3.5) will be infinite, but at every degree $(M; N)$ only a finite number of spaces contribute.

Comparing the results of the two different computations of $h_{\widehat{V}}(y; q)$ will produce the required q -identity.

In section 3.2 we apply this idea to the example discussed in section 2.2 and show that this leads to the identity (1.1) alluded to in the introduction. Other examples based on quadratic monomial ideals will be discussed in sections 4 and 5 of this paper.

In other cases we might have additional information on $\mathbf{S}(\widehat{V})$, e.g., it can be that $\mathbf{S}(\widehat{V})$ admits the action of a (Lie) algebra, in which case $h_{\widehat{V}}(y; q)$ can actually be interpreted as a character of this algebra, which might be known independently. This will be a particularly useful point of view in the case of flag varieties and will be the subject of a future publication [BH].

3.2. Prime example: $I = \langle x_1 x_2 \rangle$.

Consider again the projective variety V defined by the ideal $I = \langle x_1 x_2 \rangle \subset \mathbb{C}[x_1, x_2]$ (cf. section 2.2). The affinized variety \widehat{V} is defined by the ideal $\widehat{I} \subset \mathbb{C}[x_1[m], x_2[m]] = \widehat{\mathbf{S}}$ generated by all $f[m], m \in \mathbb{Z}_{\geq 0}$, where

$$(3.6) \quad f[m] = (x_1 x_2)[m] = \sum_{r+s=m} x_1[r] x_2[s].$$

We have a multi-degree on $\mathbf{S}(\widehat{V}) = \widehat{\mathbf{S}}/\widehat{I}$ defined by

$$(3.7) \quad \deg(x_1[m]) = (1, 0; m), \quad \deg(x_2[m]) = (0, 1; m).$$

Using the relations (3.6), it can be shown that a basis for $\mathbf{S}(\widehat{V})_{(M_1, M_2)}$ is given by

$$(3.8) \quad \begin{aligned} & x_1[n_{M_1}^{(1)}] \dots x_1[n_1^{(1)}] x_2[n_{M_2}^{(2)}] \dots x_2[n_1^{(2)}] \\ \text{with } & n_{M_1}^{(1)} \geq \dots \geq n_1^{(1)} \geq M_2 \quad \text{and} \quad n_{M_2}^{(2)} \geq \dots \geq n_1^{(2)} \geq 0. \end{aligned}$$

Before we prove (3.8), let us notice that by using

$$(3.9) \quad \sum_{n_1 \geq \dots \geq n_m \geq 0} q^{n_1 + \dots + n_m} = \frac{1}{(q)_m},$$

we immediately find

$$(3.10) \quad h_{\widehat{V}}(M_1, M_2; q) = \frac{q^{M_1 M_2}}{(q)_{M_1} (q)_{M_2}}.$$

Proof of (3.8). To prove the claim, we first have to show that every monomial

$$(3.11) \quad x_1[n_{M_1}^{(1)}] \dots x_1[n_1^{(1)}] x_2[n_{M_2}^{(2)}] \dots x_2[n_1^{(2)}],$$

with $n_{M_1}^{(1)} \geq \dots \geq n_1^{(1)} \geq 0$ and $n_{M_2}^{(2)} \geq \dots \geq n_1^{(2)} \geq 0$, can be written as a linear combination of monomials (3.8) modulo terms in the ideal \widehat{I} generated by the $f[m]$. Clearly, it suffices to prove this for $M_1 = 1$ and $n_1^{(1)} = M_2 - 1$. First, we claim that

$$(3.12) \quad x_1[k] x_2[n_{M_2}^{(2)}] \dots x_2[n_1^{(2)}] \in \widehat{I} \quad \forall k \leq M_2 - 1, M_2 \geq 1$$

This is proved by induction to M_2 . Obviously, for $M_2 = 1$, $x_1[0]x_2[0] = f[0] \in \widehat{I}$. The induction step $M_2 \rightarrow M_2 + 1$ follows from

$$x_1[M_2]x_2[0]^{M_2+1} \sim - \sum_{k=1}^{M_2} x_1[M_2-k]x_2[k]x_2[0]^{M_2-1} \in \widehat{I},$$

where in the last step we have used the induction hypothesis and by \sim we denote equivalence upto terms in the ideal \widehat{I} . Now, let d denote the sum of the $(M_2 - 1)$ -st smallest arguments of the x_2 -variables in the monomial (3.11), i.e., $d = \sum_{j=1}^{M_2-1} n_j^{(2)}$. We will prove, by a nested induction to (M_2, d) , that each monomial

$$x_1[M_2 - 1]x_2[n_{M_2}^{(2)}] \dots x_2[n_1^{(2)}],$$

with $n_{M_2}^{(2)} \geq \dots \geq n_1^{(2)} \geq 0$, can be written in the form (3.8) modulo terms in the ideal. Denote by \mathcal{M} the span of (3.8). For $d = 0$ and arbitrary $M_2 \geq 1$ we have, using (3.12),

$$\begin{aligned} x_1[M_2 - 1]x_2[m]x_2[0]^{M_2-1} &\sim - \sum_{\substack{k=0, \dots, M_2-1+m \\ k \neq m}} x_1[M_2 - 1 + m - k]x_2[k]x_2[0]^{M_2-1} \\ &\sim - \sum_{k=0, \dots, m-1} x_1[M_2 - 1 + m - k]x_2[k]x_2[0]^{M_2-1} \in \mathcal{M}, \end{aligned}$$

where $m \in \mathbb{Z}_{\geq 0}$ is arbitrary. Now, for the induction step $(M_2, d) \rightarrow (M_2, d + 1)$, assume the statement is true for all $M'_2 = M_2, d' \leq d$ and $M'_2 < M_2$, all d' . Consider

$$x_1[M_2 - 1]x_2[n_{M_2}^{(2)}] \dots x_2[n_1^{(2)}],$$

with $n_{M_2}^{(2)} \geq \dots \geq n_1^{(2)} \geq 0$ and $\sum_{j=1}^{M_2-1} n_j^{(2)} = d + 1$. Omitting terms in \mathcal{M} we have

$$x_1[M_2 - 1]x_2[n_{M_2}^{(2)}] \dots x_2[n_1^{(2)}] \sim \sum_{k=1}^{M_2-1} x_1[M_2 - 1 - k]x_2[n_{M_2}^{(2)} + k]x_2[n_{M_2-1}^{(2)}] \dots x_2[n_1^{(2)}].$$

By the induction step we can write

$$\begin{aligned} &x_1[M_2 - 1 - k]x_2[n_{M_2}^{(2)} + k]x_2[n_{M_2-1}^{(2)}] \dots x_2[n_1^{(2)}] \\ &\sim \sum_{m_1 \leq \dots \leq m_{M_2-1}} a_{m_1 \dots m_{M_2-1}}^{(k)} x_1[M_2 - 1]x_2[n_{M_2}^{(2)} + k]x_2[m_{M_2-1}] \dots x_2[m_1], \end{aligned}$$

again modulo terms in \mathcal{M} . But now notice that in

$$\sum_{k=1}^{M_2-1} \sum_{m_1 \leq \dots \leq m_{M_2-1}} a_{m_1 \dots m_{M_2-1}}^{(k)} x_1[M_2 - 1]x_2[n_{M_2}^{(2)} + k]x_2[m_{M_2-1}] \dots x_2[m_1],$$

all terms have $d' \leq d$ and hence are in \mathcal{M} (modulo \widehat{I}) by the induction hypothesis. The proof is completed once we show the converse statement, i.e., that no

linear combination of monomials (3.8) is in the ideal \widehat{I} . This is proved similarly as above. \square

Remark 3.3. As an aside, we remark that the monomials (3.8) are the complement of the leading terms of a Gröbner basis for I with respect to the lexicographic order defined by $x_1[0] > x_1[1] > \dots > x_2[0] > x_2[1] > \dots$. The Gröbner basis can in principle be found by applying Buchberger's algorithm (see, e.g., [CLO1,CLO2]).

On the other hand, a resolution of $\mathbf{S}(\widehat{V})$ is easily constructed since in this case \widehat{V} is a complete intersection. To this end, let \mathcal{I}_m be the set of ordered m -tuples $\{n_1, \dots, n_m\}$ with $n_1 > \dots > n_m \geq 0$, and let $F^{(m)}$ be the free $\widehat{\mathbf{S}}$ -module on generators e_I , $I \in \mathcal{I}_m$, with $\deg(e_I) = (-m, -m; -n_1 - \dots - n_m)$. I.e.,

$$(3.13) \quad F^{(1)} \cong \bigoplus_{n \geq 0} \widehat{\mathbf{S}}(-1, -1; -n), \quad F^{(m)} \cong \bigwedge^m F^{(1)}.$$

For $I = \{n_1, \dots, n_m\} \in \mathcal{I}_m$, and $J \in \mathcal{I}_{m-1}$ define

$$(3.14) \quad c_{IJ} = \begin{cases} 0 & \text{if } J \not\subset I, \\ (-1)^k f[n_k] & \text{if } I = J \cup \{n_k\}. \end{cases}$$

We have maps $d_m : F^{(m)} \rightarrow F^{(m-1)}$, satisfying $d_{m-1}d_m = 0$, defined by

$$(3.15) \quad d_m : e_I \mapsto \sum_{J \in \mathcal{I}_{m-1}} c_{IJ} e_J,$$

and such that the complex

$$(3.16) \quad \dots \xrightarrow{d_3} F^{(2)} \xrightarrow{d_2} F^{(1)} \xrightarrow{d_1} \widehat{\mathbf{S}} \rightarrow \mathbf{S}(\widehat{V}) \rightarrow 0,$$

is exact, i.e., provides a resolution of $\mathbf{S}(\widehat{V})$. Now, clearly,

$$\begin{aligned} h_{F^{(m)}}(M_1, M_2; q) &= \sum_{n_1 > \dots > n_m \geq 0} \sum_{N \geq 0} \dim \mathbf{S}(-m, -m; -n_1 - \dots - n_m)_{(M_1, M_2; N)} q^N \\ &= \sum_{n_1 > \dots > n_m \geq 0} \sum_{N \geq 0} \dim \mathbf{S}_{(M_1-m, M_2-m; N-n_1-\dots-n_m)} q^N \\ &= \sum_{n_1 > \dots > n_m \geq 0} q^{n_1+\dots+n_m} \sum_{N \geq 0} \dim \mathbf{S}_{(M_1-m, M_2-m; N)} q^N \\ (3.17) \quad &= \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} \frac{1}{(q)_{M_1-m}(q)_{M_2-m}}, \end{aligned}$$

where we have used (3.9). Thus, by applying the Euler-Poincaré principle to the resolution (3.16), we find

$$(3.18) \quad h_{\widehat{V}}(M_1, M_2; q) = \sum_{m \geq 0} (-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} \frac{1}{(q)_{M_1-m}(q)_{M_2-m}}.$$

Equating the expressions (3.10) and (3.18) leads to the q -identity (1.1). This identity first occurred in [PS1, PS2] where it was used to compare two different

quasi-particle descriptions of the $(\widehat{\mathfrak{sl}_3})_1$ modules (see also [BH]). It was proved in [BS2] by generating function techniques.

The (full) Hilbert series (3.3) follows straightforwardly from either (3.10) or (3.18)

$$(3.19) \quad h_{\widehat{V}}(y; q) = \frac{(y_1 y_2; q)_\infty}{(y_1; q)_\infty (y_2; q)_\infty},$$

where

$$(3.20) \quad (y; q)_N = \prod_{k=1}^N (1 - yq^{k-1}).$$

Remark 3.4. Instead of considering the affinized projective variety \widehat{V} , one may also consider a partial affinization V_N defined as the (finite dimensional) projective variety associated to the ideal $I_N \subset \mathbf{S}_N$, where $\mathbf{S}_N = \mathbb{C}[x_1[m], x_2[m]]_{0 \leq m \leq N}$ and $I_N = \langle f[m] \rangle_{0 \leq m \leq N}$. While an explicit monomial basis of $\mathbf{S}(V_N) = \mathbf{S}_N/I_N$, analogous to (3.8), is considerably more complicated than in the fully affinized case, the resolution of $\mathbf{S}(V_N)$ is simply the restriction of the resolution (3.16) to all m -tuples $\{n_1, \dots, n_m\}$ satisfying $N \geq n_1 > \dots > n_m \geq 0$. Using the analogue of (3.9)

$$(3.21) \quad \sum_{N \geq n_1 \geq \dots \geq n_m \geq 0} q^{n_1 + \dots + n_m} = \begin{bmatrix} N + m \\ m \end{bmatrix},$$

where

$$(3.22) \quad \begin{bmatrix} m \\ n \end{bmatrix} = \frac{(q)_m}{(q)_n (q)_{m-n}},$$

denotes the Gaussian polynomial, we find the Hilbert series

$$(3.23) \quad h_{V_N}(M_1, M_2; q) = \sum_{m \geq 0} (-1)^m q^{\frac{1}{2}m(m-1)} \begin{bmatrix} N + 1 \\ m \end{bmatrix} \begin{bmatrix} N + M_1 - m \\ M_1 - m \end{bmatrix} \begin{bmatrix} N + M_2 - m \\ M_2 - m \end{bmatrix},$$

or, equivalently,

$$(3.24) \quad h_{V_N}(y; q) = \frac{(y_1 y_2; q)_{N+1}}{(y_1; q)_{N+1} (y_2; q)_{N+1}}.$$

4. QUADRATIC MONOMIAL IDEALS

In this section we illustrate the procedure outlined in section 3 by discussing some examples of q -identities associated to projective varieties with monomial quadratic defining relations.

4.1. A basis of $\mathbf{S}(\widehat{V})$.

Let \mathcal{P} be a set of (ordered) pairs (i, j) , $i < j$, with $i, j \in \{1, \dots, n\}$. Consider the quadratic monomial ideal $I = \langle x_i x_j \rangle_{(i,j) \in \mathcal{P}} \subset \mathbb{C}[x_1, \dots, x_n] \equiv \mathbf{S}$ with associated projective variety $V = \mathbf{V}(I)$.

We have a multi-degree $M = (M_1, \dots, M_n)$ on $\mathbf{S}(V) = \mathbb{C}[x_1, \dots, x_n]/I$, where M_i is the number of x_i in a monomial x^α . Let \widehat{V} be the affinization of V as defined in section 2.

Theorem. *A basis of $\mathbf{S}(\widehat{V})$ is given by the following monomials*

$$(4.1) \quad x_1[n_{M_1}^{(1)}] \dots x_1[n_1^{(1)}] x_2[n_{M_2}^{(2)}] \dots x_2[n_1^{(2)}] \dots x_n[n_{M_n}^{(n)}] \dots x_n[n_1^{(n)}],$$

with

$$(4.2) \quad n_{M_i}^{(i)} \geq \dots \geq n_2^{(i)} \geq n_1^{(i)} \geq \sum_{(i,j) \in \mathcal{P}} M_j, \quad \forall i.$$

We will omit the proof, which is a straightforward generalization of the proof in section 3.2.

Remark 4.1. Note that the basis of $\mathbf{S}(V) \subset \mathbf{S}(\widehat{V})$ under the identification $x_i \sim x_i[0]$ induced by (4.1) is the obvious one. A monomial $x_{i_1} \dots x_{i_t} \in \mathbf{S}(V)$ iff there is no pair (i_r, i_s) such that $(i_r, i_s) \in \mathcal{P}$.

Using the basis (4.1), it immediately follows that the (partial) Hilbert series of \widehat{V} is given by (cf. (3.10))

$$(4.3) \quad h_{\widehat{V}}(M_1, \dots, M_n; q) = \frac{q^{\sum_{(i,j) \in \mathcal{P}} M_i M_j}}{(q)_{M_1} \dots (q)_{M_n}}.$$

4.2. q -identities; the algorithm.

Having found the Hilbert series of a general affinized projective variety \widehat{V} corresponding to the affinization \widehat{I} of a quadratic monomial ideal I , we can now, in principle, obtain a q -identity by explicitly constructing a free resolution of the coordinate ring $\mathbf{S}(\widehat{V})$ of \widehat{V} as we have done in the example of section 3.2. In this paper, however, we will take a different approach and ‘construct’ a q -identity by repeatedly using the basic identity (1.1) with the underlying resolution of $\mathbf{S}(V)$ (section 2.3) as a guiding principle.

Conjecturally, the resulting alternating sum formula will also arise by applying the Euler-Poincaré principle to a certain resolution of $\mathbf{S}(\widehat{V})$ which, in some sense, is an appropriately ‘affinized’ version of Taylor’s resolution of $\mathbf{S}(V)$.

In the remainder of this section we will explain the algorithm and some of the properties of the resulting q -identity. In section 5 we will illustrate the algorithm in a few examples.

Consider the expression

$$(4.4) \quad \frac{q^{\sum_{(i,j) \in \mathcal{P}} M_i M_j}}{(q)_{M_1} \dots (q)_{M_n}}.$$

We now construct an alternating sum formula, bearing close resemblance to the Taylor resolution, as follows

- Order the quadratic monomials $x_i x_j$, $(i, j) \in \mathcal{P}$, in any arbitrary way $\{f_1 = x_{i_1} x_{j_1}, \dots, f_t = x_{i_t} x_{j_t}\}$. Then, apply (1.1) to the term

$$\frac{q^{M_{i_1} M_{j_1}}}{\binom{(\cdot)}{(\cdot)} \binom{(\cdot)}{(\cdot)}}$$

in (4.4), calling the summation variable m_1 . We proceed with the term in (4.4) corresponding to f_2 . If $f_{12} = f_1 f_2$ (cf. section 2.3), we can apply (1.1) immediately. On the other hand, if $f_{12} \neq f_1 f_2$ then one of the variables M_{i_1} or M_{j_1} appears in $M_{i_2} M_{j_2}$, and the corresponding term in the denominator will have been shifted by m_1 . Making the corresponding shift in the exponent, i.e., writing $MM' = (M - m_1)M' + m_1 M'$, we can apply (1.1) to the $(M - m_1)M'$ part, denoting the summation variable by m_2 . We continue this process until all terms $M_i M_j$, $(i, j) \in \mathcal{P}$, in the exponent of (4.4) have been replaced. The resulting expression will be an alternating sum with summation variables m_k , $k = 1, \dots, t$, in 1–1 correspondence with the generating monomials $f_k = x_{i_k} x_{j_k}$. The M_i dependent remnant in the q -exponent will be of the form

$$(4.5) \quad \sum_{\substack{I \in \mathcal{I}_2 \\ I = \{k\} \cup \{l\}}} d_I m_k M_{j_l},$$

where $d_I = 0$ if $f_{kl} = f_k f_l$, and $d_I = 1$ if $f_{kl} \neq f_k f_l$.

- In the second step we repeat the procedure to the monomials $m_k M_{j_l}$, incorporating the appropriate shifts in the m_k and M_{j_l} , and calling the corresponding summation variables m_{kl} . Clearly, the summation variables introduced in this step are in 1–1 correspondence with the f_I , $I \in \mathcal{I}_2$, such that $f_{kl} \neq f_k f_l$. Note that, in (4.5), it can happen that $j_l = j_{l'}$ for some $l \neq l'$ (cf. section 5.4 for an example). In that case it is important to keep the terms separate and remember their origin. The M_i dependent remnant in the q -exponent will now be of the form

$$(4.6) \quad \sum_{\substack{I \in \mathcal{I}_3 \\ I = \{k, l\} \cup \{m\}}} d_I m_{kl} M_{j_m},$$

where $d_I = 0$ if $f_{klm} = f_{kl} f_m$ and 1 otherwise.

- Continue the procedure as before until all M_i dependent parts in the q -exponent have been replaced by alternating sums.
- As a last step we shift all the summation variables m_I such that they appear in the denominator as $\prod_I (q)_{m_I}$.

The resulting identity will be of the form

$$(4.7) \quad \frac{q^{\sum_{(i,j) \in \mathcal{P}} M_i M_j}}{(q)_{M_1} \cdots (q)_{M_n}} = \sum_{m_I \geq 0, I \in \mathcal{I}'} (-1)^{\sum_{I \in \mathcal{I}'} |I| m_I} \frac{q^{Q(m_I)}}{\prod_{I \in \mathcal{I}'} (q)_{m_I} \prod_i (q)_{M_i - \Delta M_i}},$$

where \mathcal{I}' is the subset of \mathcal{I} consisting of all sets $\{i_1, \dots, i_s\}$ such that $f_{i_1, \dots, i_s} \neq f_{i_1, \dots, \hat{i}_r, \dots, i_s} f_{i_r}$ for some $1 \leq r \leq s$. Furthermore,

$$(4.8) \quad \Delta M_i = \sum_{I \in \mathcal{I}'} a_I^{(i)} m_I,$$

where the $a_I^{(i)}$ are a set of positive integers such that $a_I^{(i)} \neq 0$ iff x_i occurs in f_I , and

$$(4.9) \quad Q(m_I) = \frac{1}{2} \sum_{I \in \mathcal{I}'} |I| m_I (m_I - 1) + Q'(m_I)$$

for some positive definite bilinear form Q' .

Some more features of the expression (4.7) – (4.9) can be derived by examining how the Hilbert function of the underlying variety V is reproduced. To this end it is convenient to multiply both sides by $\prod_i y_i^{M_i}$ and sum over $M_i \geq 0$ (cf. the discussion in section 1). On the left hand side of (4.7), the $\mathcal{O}(q^0)$ -terms obviously correspond to a basis for $\mathbf{S}(V)$ (cf. remark 4.1). On the right hand side we get contributions only from $m_I = 0$ or $m_I = 1$. If $m_I = 0$ for all I we find a contribution

$$(4.10) \quad \prod_{i=1}^n \frac{1}{(1 - y_i)},$$

while if $m_I = 1$, and $m_J = 0$ for all $J \neq I$ contributes, the contribution will be

$$(4.11) \quad (-1)^{|I|} \prod_{i=1}^n \frac{y_i^{a_I^{(i)}}}{(1 - y_i)},$$

to be compared to (2.10). If all $I \in \mathcal{I}$ would contribute to the right hand side of (4.7) through (4.11), then we would get exactly the expression (2.10) corresponding to Taylor's resolution (cf. examples 5.1 and 5.4). However, the sum in (4.7) is over $I \in \mathcal{I}' \subset \mathcal{I}$ and in general $\mathcal{I}' \neq \mathcal{I}$. The ‘missing terms’ in (2.10) are recovered as follows. Suppose $I = \{i_1, \dots, i_s\} = J \cup \{i_r\}$ for some $1 \leq r \leq s$, and such that $f_I = f_J f_{i_r}$, i.e., $I \notin \mathcal{I}'$. Then the positive definite bilinear form $Q'(m_I)$ in (4.9) will not contain a term $m_J m_{i_r}$. In other words, the term in the summation on the right hand side of (4.7) with $m_J = 1$, $m_{i_r} = 1$ and $m_I = 0$, for all other I , will contribute to the $\mathcal{O}(q^0)$ -term. The contribution is exactly (cf. example 5.2)

$$(4.12) \quad (-1)(-1)^{|J|} \prod_{i=1}^n \frac{y_i^{a_J^{(i)} + a_{i_r}^{(i)}}}{(1 - y_i)} = (-1)^{|I|} \prod_{i=1}^n \frac{y_i^{a_I^{(i)}}}{(1 - y_i)}.$$

This is one way the affinized expression (4.7) ‘knows about’ Koszul parts in the Taylor resolution (cf. remark 2.1) and automatically takes care of them without having to introduce, in some sense trivial, additional summation variables.

It may also happen that the bilinear form $Q'(m_I)$ in (4.9) still contains quadratic pieces m_I^2 for some I . In that case the term with $m_I = 1$ and $m_J = 0$, for all other $J \in \mathcal{I}'$, will not contribute to the $\mathcal{O}(q^0)$ -term on the right hand side of (4.7). This will only happen if there exists another $I' \in \mathcal{I}$, for which the same thing happens and for which $f_I = f_{I'}$, $|I| = -|I'|$, i.e., in that case the contributions from I and I' in Taylor's resolution will cancel (cf. example 5.4). Whenever this happens it might indicate that Taylor's resolution is not a minimal resolution and that it can be reduced by removing the spaces corresponding to I and I' (cf. the example in section 2.3).

If in Taylor's resolution there exists an $I \in \mathcal{I}$ such that $f_I = f_J f_{J'}$ for some $J \cup J' \subset I$, this is another indication that the resolution might not be a minimal one. In that case, in the affinized expression (4.7), it might be possible to explicitly sum out the summation variable m_I to obtain an expression associated to some reduction of Taylor's resolution (cf. example 5.2). This is another way in which the affinized expression (4.7) knows about Koszul parts in Taylor's resolution.

Finally, it should be obvious that the final form of the identity (4.7) is not necessarily unique but could depend on the order in which the various summation variables m_I are introduced. In principle we could fix the expression by specifying the order of m_I (e.g., through a reverse graded lexicographic ordering on the I), but in practise the identities are more easily accessible by using already established identities for sub-ideals as ‘building blocks’ (cf. examples 5.3 and 5.5). Also, it might very well be that specific identities are more ‘manageable’ or ‘useful’ than others.

5. EXAMPLES

In this section we will illustrate the algorithm outlined in section 4.2 and some of the properties of the resulting q -identities by explicitly going through a few examples of quadratic monomial ideals. The main results are the identities (5.2), (5.6), (5.10), (5.13), (5.19) and (5.23). The results can be used as building blocks for more complicated examples.

5.1. $I = \langle x_1x_2, x_2x_3 \rangle$.

Consider the ideal $I = \langle x_1x_2, x_2x_3 \rangle \subset \mathbb{C}[x_1, x_2, x_3]$. The corresponding variety $\mathbf{V}(I)$ is a 1-dimensional subvariety of \mathbb{P}^2 (union of a line and a point). Taylor’s resolution (2.17) of $\mathbf{S}(V)$ takes the form

$$(5.1) \quad 0 \rightarrow \mathbf{S}(-1, -1, -1) \rightarrow \mathbf{S}(-1, -1, 0) \oplus \mathbf{S}(0, -1, -1) \rightarrow \mathbf{S} \rightarrow \mathbf{S}(V) \rightarrow 0.$$

By repeated application of the basic identity (1.1), following the algorithm outlined in section 4.2, we find

$$\begin{aligned}
 & \frac{q^{M_1 M_2 + M_2 M_3}}{(q)_{M_1} (q)_{M_2} (q)_{M_3}} \\
 &= \sum_{m_1} (-1)^{m_1} \frac{q^{\frac{1}{2} m_1 (m_1 - 1)}}{(q)_{m_1}} \frac{q^{M_2 M_3}}{(q)_{M_1 - m_1} (q)_{M_2 - m_1} (q)_{M_3}} \\
 &= \sum_{m_1} (-1)^{m_1} \frac{q^{\frac{1}{2} m_1 (m_1 - 1)}}{(q)_{m_1}} \frac{q^{(M_2 - m_1) M_3 + m_1 M_3}}{(q)_{M_1 - m_1} (q)_{M_2 - m_1} (q)_{M_3}} \\
 &= \sum_{m_1, m_2} (-1)^{m_1 + m_2} \frac{q^{\frac{1}{2} m_1 (m_1 - 1) + \frac{1}{2} m_1 (m_1 - 1)}}{(q)_{m_1} (q)_{m_2}} \frac{q^{m_1 M_3}}{(q)_{M_1 - m_1} (q)_{M_2 - (m_1 + m_2)} (q)_{M_3 - m_2}} \\
 &= \sum_{m_1, m_2} (-1)^{m_1 + m_2} \frac{q^{\frac{1}{2} m_1 (m_1 - 1) + \frac{1}{2} m_1 (m_1 - 1)}}{(q)_{m_1} (q)_{m_2}} \frac{q^{m_1 (M_3 - m_2) + m_1 m_2}}{(q)_{M_1 - m_1} (q)_{M_2 - (m_1 + m_2)} (q)_{M_3 - m_2}} \\
 &= \sum_{m_1, m_2, m_{12}} (-1)^{m_1 + m_2 + m_{12}} \frac{q^{\frac{1}{2} m_1 (m_1 - 1) + \frac{1}{2} m_1 (m_1 - 1) + \frac{1}{2} m_{12} (m_{12} - 1)}}{(q)_{m_1 - m_{12}} (q)_{m_2} (q)_{m_{12}}} \\
 &\quad \times \frac{q^{m_1 m_2}}{(q)_{M_1 - m_1} (q)_{M_2 - (m_1 + m_2)} (q)_{M_3 - (m_2 + m_{12})}} \\
 &= \sum_{m_1, m_2, m_{12}} (-1)^{m_1 + m_2} \frac{q^{\frac{1}{2} m_1 (m_1 - 1) + \frac{1}{2} m_2 (m_2 - 1) + m_{12} (m_{12} - 1)}}{(q)_{m_1} (q)_{m_2} (q)_{m_{12}}} \\
 &\quad \times \frac{q^{m_1 m_2 + m_{12} (m_1 + m_2)}}{(q)_{M_1 - m_1} (q)_{M_2 - (m_1 + m_2)} (q)_{M_3 - (m_2 + m_{12})}},
 \end{aligned}
 \tag{5.2}$$

where, in the last step, we have shifted the summation variable $m_1 \rightarrow m_1 + m_{12}$.

Indeed, (5.2) is of the form (4.7) with $\mathcal{I}' = \mathcal{I} = \{1, 2, 12\}$,

$$(5.3) \quad \Delta M_1 = m_1 + m_{12}, \quad \Delta M_2 = m_1 + m_2 + m_{12}, \quad \Delta M_3 = m_2 + m_{12},$$

and

$$(5.4) \quad Q'(m_I) = m_1 m_2 + m_{12}(m_1 + m_2).$$

The $\mathcal{O}(q^0)$ -term in the resulting identity for the Hilbert series $h_{\hat{V}}(y; q)$ leads to the identity

$$(5.5) \quad \frac{1}{(1-y_1)(1-y_3)} + \frac{y_2}{1-y_2} = \frac{1-y_1 y_2 - y_2 y_3 + y_1 y_2 y_3}{(1-y_1)(1-y_2)(1-y_3)}.$$

5.2. $I = \langle x_1 x_2, x_2 x_3, x_3 x_4 \rangle$.

Consider the ideal $I = \langle x_1 x_2, x_2 x_3, x_3 x_4 \rangle \subset \mathbb{C}[x_1, x_2, x_3, x_4]$. The resolution of $\mathbf{S}(V)$ was discussed in (2.19). Applying (5.2) we find

$$\begin{aligned} & \frac{q^{M_1 M_2 + M_2 M_3 + M_3 M_4}}{(q)_{M_1} (q)_{M_2} (q)_{M_3} (q)_{M_4}} \\ &= \sum_{\substack{m_I \geq 0 \\ I=1,2,12}} (-1)^{m_1+m_2} \frac{q^Q}{\prod_{I=1,2,12} (q)_{m_I} \prod_{i=1}^3 (q)_{M_i - \Delta M_i}} \frac{q^{M_3 M_4}}{(q)_{M_4}}, \end{aligned}$$

where ΔM_i , $i = 1, 2, 3$, is given by (5.3) and Q by (4.9) and (5.4). Now write

$$M_3 M_4 = (M_3 - (m_2 + m_{12})) M_4 + (m_2 + m_{12}) M_4,$$

and apply (1.1) to the first term with summation variable m_3 . Then write, in the q -exponent,

$$(m_2 + m_{12}) M_4 = (m_2 + m_{12})(M_4 - m_3) + (m_2 + m_{12}) m_3,$$

and apply (1.1) to $m_{12}(M_4 - m_3)$ with summation variable m_{123} . Finally, writing

$$m_2(M_4 - m_3) = m_2(M_4 - (m_3 + m_{123})) + m_2 m_{123},$$

and applying (1.1) to $m_2(M_4 - (m_3 + m_{123}))$ with summation variable m_{23} and shifting $m_2 \rightarrow m_2 + m_{23}$ and $m_{12} \rightarrow m_{12} + m_{123}$, yields

$$(5.6) \quad \frac{q^{M_1 M_2 + M_2 M_3 + M_3 M_4}}{(q)_{M_1} (q)_{M_2} (q)_{M_3} (q)_{M_4}} = \sum_{\substack{m_I \geq 0 \\ I \in \mathcal{I}'}} (-1)^{\sum_I |I| m_I} \frac{q^{Q(m_I)}}{\prod_{I \in \mathcal{I}'} (q)_{m_I} \prod_{i=1}^4 (q)_{M_i - \Delta M_i}},$$

where $\mathcal{I}' = \{1, 2, 3, 12, 23, 123\}$,

$$\begin{aligned} \Delta M_1 &= m_1 + m_{12} + m_{123}, \\ \Delta M_2 &= m_1 + m_2 + m_{12} + m_{23} + m_{123}, \\ \Delta M_3 &= m_2 + m_3 + m_{12} + m_{23} + m_{123}, \\ \Delta M_4 &= m_1 + m_2 + m_3 + m_{12} + m_{23} + m_{123}, \end{aligned} \tag{5.7}$$

and

$$(5.8) \quad \begin{aligned} Q &= \frac{1}{2} \sum_{I \in \mathcal{I}'} |I| m_I (m_I - 1) + m_1 m_2 + m_2 m_3 + m_{12} m_{23} \\ &\quad + (m_1 + m_2 + m_3)(m_{12} + m_{23} + m_{123}) + 2(m_{12} + m_{23})m_{123}. \end{aligned}$$

Observe, indeed, that since $f_{13} = f_1 f_3$, the subset $\{1, 3\}$ is absent from \mathcal{I}' and hence the corresponding summation variable m_{13} does not occur in (5.6).

Moreover, as discussed in section 2.3, the Taylor resolution (2.19) of $\mathbf{S}(V)$ is not minimal. A minimal resolution is obtained from (2.19) by removing the spaces corresponding to $I = 13$ and $I = 123$. This manifests itself in (5.6) by the fact that the summation variable m_{123} can be summed out.

First, notice that we can get rid of the m_{123} shifts in the q -numbers in the denominator, by shifting $m_1 \rightarrow m_1 - m_{123}$ and $m_3 \rightarrow m_3 - m_{123}$. This yields an exponent

$$Q \rightarrow Q' + \frac{1}{2} m_{123} (m_{123} - 1),$$

with

$$(5.9) \quad \begin{aligned} Q' &= \frac{1}{2} \sum_{i=1,2,3} m_i (m_i - 1) + \sum_{ij=12,23} m_{ij} (m_{ij} - 1) \\ &\quad + m_1 m_2 + m_2 m_3 + (m_1 + m_2 + m_3)(m_{12} + m_{23}) + m_{12} m_{23}. \end{aligned}$$

Next, we can sum out m_{123} by (1.1) after which we obtain

$$(5.10) \quad \frac{q^{M_1 M_2 + M_2 M_3 + M_3 M_4}}{(q)_{M_1} (q)_{M_2} (q)_{M_3} (q)_{M_4}} = \sum_{\substack{m_I \geq 0 \\ I \in \mathcal{I}''}} (-1)^{\sum_I |I| m_I} \frac{q^{Q''}}{\prod_{I \in \mathcal{I}''} (q)_{m_I} \prod_{i=1}^4 (q)_{M_i - \Delta M_i}},$$

where $\mathcal{I}'' = \{1, 2, 3, 12, 23\}$,

$$(5.11) \quad \begin{aligned} \Delta M_1 &= m_1 + m_{12}, \\ \Delta M_2 &= m_1 + m_2 + m_{12} + m_{23}, \\ \Delta M_3 &= m_2 + m_3 + m_{12} + m_{23}, \\ \Delta M_4 &= m_3 + m_{23}, \end{aligned}$$

and

$$(5.12) \quad Q'' = Q' + m_1 m_3,$$

with Q' given by (5.9).

5.3. $I = \langle x_1 x_2, x_2 x_3, \dots, x_{n-1} x_n \rangle$.

The ideal $I = \langle x_1 x_2, x_2 x_3, \dots, x_{n-1} x_n \rangle \subset \mathbb{C}[x_1, \dots, x_n]$ corresponds to a dimension $\left\lfloor \frac{n-1}{2} \right\rfloor$ variety $V_n \subset \mathbb{P}^{n-1}$ and generalizes the examples of sections 3.2, 5.1 and 5.2. The corresponding q -identity can be proved by induction. The result is

$$(5.13) \quad \frac{q^{\sum_{i=1}^{n-1} M_i M_{i+1}}}{(q)_{M_1} \cdots (q)_{M_n}} = \sum_{m_1, \dots, m_{n-1}} (-1)^{\sum m_i} \frac{q^{Q(m_i, n_i)}}{\prod_i (q)_{m_i} (q)_{n_i} \prod_{i=1}^n (q)_{M_i - \Delta M_i}},$$

where

$$(5.14) \quad \begin{aligned} Q = & \frac{1}{2} \sum_{i=1}^{n-1} m_i(m_i - 1) + \sum_{i=1}^{n-2} n_i(n_i - 1) + \sum_i m_i(m_{i+1} + m_{i+2}) \\ & + \sum_{i=1}^{n-1} m_i(n_{i-2} + n_{i-1} + n_i + n_{i+1} + n_{i+2}) + \sum_{i=1}^{n-2} n_i(n_{i+1} + n_{i+2}), \end{aligned}$$

and

$$(5.15) \quad \Delta M_i = m_i + m_{i-1} + n_i + n_{i-1} + n_{i-2}.$$

For simplicity of notation we have denoted $n_i = m_{i+1}$ and $m_n \equiv m_0 \equiv n_0 \equiv n_{-1} \equiv n_{n-1} \equiv n_n \equiv 0$. Note that, for $M_j = 0$, the q -identity factorizes and reduces to the same identity for smaller n .

The induction procedure that leads to (5.13) suggests the following recursion relation for the Hilbert series $h_n(y)$ of the underlying variety V_n

$$(5.17) \quad h_n(y_1, \dots, y_n) = \frac{1}{1 - y_n} h_{n-2}(y_1, \dots, y_{n-2}) + \frac{y_{n-1}}{1 - y_{n-1}} h_{n-3}(y_1, \dots, y_{n-3}),$$

with $h_0 = 1$, $h_1 = \frac{1}{1-y_1}$ and $h_2 = \frac{1}{1-y_2} + \frac{y_1}{1-y_1}$.

5.4. $I = \langle x_1 x_2, x_2 x_3, x_1 x_3 \rangle$.

To obtain a q -identity for the ideal $I = \langle x_1 x_2, x_2 x_3, x_1 x_3 \rangle \subset \mathbb{C}[x_1, x_2, x_3]$, corresponding to three non-collinear points in \mathbb{P}^2 (cf. [Ha, Example 13.11]), we need the following lemma

$$(5.18) \quad \frac{q^{2MN}}{(q)_M (q)_N} = \sum_{r,s,t} (-1)^{r+s} \frac{q^{\frac{1}{2}r(r-1) + \frac{1}{2}s(s-1) + t(t-1) + (rs+rt+st) + (r+t)M}}{(q)_r (q)_s (q)_t (q)_{M-(r+s+t)} (q)_{N-(r+s+2t)}},$$

which is proved by the same techniques as before, i.e., write $2MN = MN + MN$ and apply (1.1) to MN with summation variable r . Then in the remaining exponent write $MN = (M-r)(N-r) + rN + rM - r^2$. Apply (1.1) again, now to $(M-r)(N-r)$ with summation variable s and write in the remaining exponent $rN = r(N - (r+s)) + r(r+s)$ and apply (1.1) to $r(N - (r+s))$ with summation variable t . Finally, shift $r \rightarrow r+t$. This yields (5.18).

Now, in

$$\frac{q^{M_1 M_2 + M_2 M_3 + M_1 M_3}}{(q)_{M_1} (q)_{M_2} (q)_{M_3}},$$

we apply (1.1) consecutively to $M_1 M_2$, $(M_2 - m_1) M_3$, $(M_1 - m_1)(M_3 - m_2)$ and $m_2 M_1$ with summation variables m_1, m_2, m_3 and m_{23} . This yields

$$\begin{aligned} & \sum_{\substack{m_I \geq 0 \\ I=1,2,3,23}} (-1)^{\sum m_I} \frac{q^Q}{(q)_{m_1} (q)_{m_2 - m_{23}} (q)_{m_3} (q)_{m_{23}}} \\ & \times \frac{1}{\frac{(-)}{(-)} \frac{(-)}{(-)} \frac{(-)}{(-)}}, \end{aligned}$$

with

$$Q = \frac{1}{2} \sum_{i=1,2,3} m_i(m_i - 1) + \frac{1}{2} m_{23}(m_{23} - 1) + m_2 m_3 + 2m_1 M_3.$$

Then write $2m_1 M_3 = 2m_1(M_3 - (m_2 + m_3))$ and apply (5.18) with the substitutions $r \rightarrow m_{13}$, $s \rightarrow m_{12}$ and $t \rightarrow m_{123}$. In the result shift $m_1 \rightarrow m_1 + m_{12} + m_{13} + m_{123}$ and $m_2 \rightarrow m_2 + m_{23}$. This finally yields

$$(5.19) \quad \frac{q^{M_1 M_2 + M_2 M_3 + M_1 M_3}}{(q)_{M_1} (q)_{M_2} (q)_{M_3}} = \sum_{\substack{m_I \geq 0 \\ I \in \mathcal{I}'}} (-1)^{\sum_I |I| m_I} \frac{q^{Q(m_I)}}{\prod_{I \in \mathcal{I}'} (q)_{m_I} \prod_{i=1}^3 (q)_{M_i - \Delta M_i}},$$

with $\mathcal{I}' = \{1, 2, 3, 12, 23, 13, 123\} = \mathcal{I}$,

$$(5.20) \quad \begin{aligned} \Delta M_1 &= m_1 + m_3 + m_{12} + m_{23} + m_{13} + m_{123}, \\ \Delta M_2 &= m_1 + m_2 + m_{12} + m_{23} + m_{13} + m_{123}, \\ \Delta M_3 &= m_2 + m_3 + m_{12} + m_{23} + m_{13} + 2m_{123}, \end{aligned}$$

and

$$(5.21) \quad \begin{aligned} Q &= \frac{1}{2} \sum_{I \in \mathcal{I}'} |I| m_I (m_I - 1) + m_{13}^2 + m_{123}^2 \\ &\quad + 2m_1(m_2 + m_3) + m_2 m_3 + 2m_{23}(m_{12} + m_{13}) + 3m_{12} m_{13} \\ &\quad + m_{12}(m_1 + 2m_2 + 2m_3) + m_{23}(2m_1 + m_2 + m_3) + m_{13}(m_1 + m_2 + m_3) \\ &\quad + 2m_{123}(m_1 + m_2 + m_3) + m_{123}(3m_{12} + 2m_{23} + 4m_{13}). \end{aligned}$$

The $\mathcal{O}(q^0)$ -term in the resulting identity for the Hilbert series $h_{\widehat{V}}(y; q)$ leads to the identity

$$(5.22) \quad \frac{1}{1 - y_1} + \frac{y_2}{1 - y_2} + \frac{y_3}{1 - y_3} = \frac{1 - y_1 y_2 - y_2 y_3 - y_1 y_3 + 2y_1 y_2 y_3}{(1 - y_1)(1 - y_2)(1 - y_3)}.$$

Note that in deriving (5.22) the terms $m_I = 1$ and all others vanishing, do not contribute for $I = 13$ and $I = 123$ due to the terms m_I^2 in (5.21). Indeed, since $f_{13} = f_{123}$ their contribution to $h_V(y)$ in (2.10) would cancel. This is related to the fact that, also in this case, Taylor's resolution is not minimal but can be further reduced by omitting the spaces corresponding to $I = 13$ and $I = 123$. So, even though it does not seem possible to further simplify the affine identity (5.19), the identity somehow knows about the non-minimality of Taylor's resolution.

5.5. $I = \langle x_1 x_2, x_2 x_3, \dots, x_{n-1} x_n, x_1 x_n \rangle$.

As a generalization of the example in section 5.4, consider the ideal $I = \langle x_1 x_2, x_2 x_3, \dots, x_{n-1} x_n, x_1 x_n \rangle \subset \mathbb{C}[x_1, \dots, x_n]$, $n \geq 6$. Using, as an intermediate step, the result (5.13) we straightforwardly find

$$(5.23) \quad \frac{q^{M_1 M_2 + \dots + M_{n-1} M_n + M_1 M_n}}{\prod_{i=1}^n (q)_{M_i}} = \sum_{\substack{m_1, \dots, m_n \\ n_1, \dots, n_n}} (-1)^{\sum m_i + p + p'} \frac{q^Q}{\prod_i (q)_{m_i} (q)_{n_i} (q)_p (q)_{p'} (q)_{M_i - \Delta M_i}},$$

with¹

$$\begin{aligned}
 \Delta M_1 &= m_1 + m_n + n_1 + n_n + n_{n-1} + p + p', \\
 \Delta M_2 &= m_2 + m_1 + n_2 + n_1 + n_n + p, \\
 \Delta M_3 &= m_3 + m_2 + n_3 + n_2 + n_1 + p, \\
 \Delta M_i &= m_i + m_{i-1} + n_i + n_{i-1} + n_{i-2}, \quad 4 \leq i \leq n-3, \\
 \Delta M_{n-2} &= m_{n-2} + m_{n-3} + n_{n-2} + n_{n-3} + n_{n-4} + p', \\
 \Delta M_{n-1} &= m_{n-1} + m_{n-2} + n_{n-1} + n_{n-2} + n_{n-3} + p', \\
 (5.24) \quad \Delta M_n &= m_n + m_{n-1} + n_n + n_{n-1} + n_{n-2} + p + p',
 \end{aligned}$$

where the subscripts on m_i and n_i have to be taken modulo n . In addition to the notation in section 5.3, we have denoted $n_{n-1} = m_{n-1}n$, $n_n = m_1n$, $p = m_12n$ and $p' = m_{n-2}n_{-1}n$. The explicit expression for $Q(m_i, n_i, p, p')$ in (5.23) is left as an exercise to the reader.

6. CONCLUDING REMARKS

In this paper we have explained an algorithm to associate a q -identity to an arbitrary projective variety V defined by a quadratic monomial ideal. The identities were argued to correspond to two different ways of computing the Hilbert series of a suitable ‘affinization’ \widehat{V} of the variety V , on the one hand by computing an explicit basis for the coordinate ring $\mathbf{S}(\widehat{V})$, on the other hand by constructing a free resolution of this coordinate ring. The algorithm was illustrated in numerous examples.

The algorithm is based on Taylor’s resolution for the coordinate ring $\mathbf{S}(V)$ of the underlying (finite-dimensional) projective variety V . This resolution is not always a minimal free resolution. In section 5.2 we have seen an example where the identity can be further reduced to an identity which one would like to associate with the minimal resolution in that case. This further reduction can typically be done for the ‘Koszul parts’ in Taylor’s resolution. In other cases, such as in the example of section 5.4, a further reduction does not appear to be possible even though Taylor’s resolution is not minimal. In that example, i.e., $I = \langle x_1x_2, x_2x_3, x_1x_3 \rangle \subset \mathbb{C}[x_1, x_2, x_3]$, the minimal resolution of $\mathbf{S}(V)$ looks like

$$(6.1) \quad 0 \rightarrow \mathbf{S}(-3)^2 \rightarrow \mathbf{S}(-2)^3 \rightarrow \mathbf{S} \rightarrow \mathbf{S}(V) \rightarrow 0.$$

[Here we have only indicated the shift in total degree.] Surprisingly, however, one can find a q -identity which one would like to associate to (6.1). It reads

$$(6.2) \quad \frac{q^{M_1M_3}}{(q)_{M_1}(q)_{M_2}(q)_{M_3}} = \sum_{\substack{m_I \geq 0 \\ I=1,2,3,12,23}} (-1)^{\sum_i m_i} \frac{q^{Q(m_I)}}{\prod_I (q)_{m_I}} \frac{1}{\prod_i (q)_{M_i - \Delta M_i}},$$

where

$$\begin{aligned}
 \Delta M_1 &= m_1 + m_2 + m_{12}, \\
 \Delta M_2 &= m_1 + m_3 + m_{12} + m_{23}, \\
 (6.3) \quad \Delta M_3 &= m_2 + m_3 + m_{23},
 \end{aligned}$$

¹For $n=4, 5$ the above formulae can be checked by direct computation of the Hilbert series.

and

$$(6.4) \quad \begin{aligned} Q(m_I) = & \frac{1}{2} \sum_{i=1,2,3} m_i(m_i - 1) + \sum_{I=12,23} m_I(m_I - 1) + m_1m_2 + m_1m_3 + m_2m_3 \\ & + (m_1 + m_2 + m_3)(m_{12} + m_{23}) + m_{12}m_{23}. \end{aligned}$$

The relation of (6.2) to the coordinate ring of $\mathbf{S}(\widehat{V})$ is however not clear to me at present.

Even though we have restricted our attention to varieties defined by quadratic monomial ideals, the idea is far more general. Indeed, one can often find interesting identities associated to more general ideals. Consider, e.g., the ‘trivial’ example of $I = \langle x \rangle \subset \mathbb{C}[x]$. Obviously, both $\mathbf{S}(V)$ and $\mathbf{S}(\widehat{V})$ only contain the constant polynomials. The resolution of $\mathbf{S}(\widehat{V})$ gives rise, however, to the not completely trivial (but well-known) q -identity

$$(6.5) \quad \sum_{m \geq 0} (-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m (q)_{M-m}} = \delta_{M,0}.$$

For a slightly less trivial example, consider the ideal $I = \langle x_1, x_2x_3(x_2 - x_3) \rangle \subset \mathbb{C}[x_1, x_2, x_3]$ corresponding to complete intersection of a line and a cubic, i.e., three collinear points in \mathbb{P}^2 (cf. [Ha, Example 13.11]). The resolution of $\mathbf{S}(V)$ is Koszul’s resolution

$$(6.6) \quad 0 \rightarrow \mathbf{S}(-4) \rightarrow \mathbf{S}(-1) \oplus \mathbf{S}(-3) \rightarrow \mathbf{S} \rightarrow \mathbf{S}(V) \rightarrow 0,$$

and the associated q -identity is

$$(6.7) \quad \sum_{\substack{m_I \geq 0 \\ I=1,2,12}} (-1)^{m_1+m_2} \frac{q^{Q(m_I)}}{\prod_I (q)_{m_I}} \frac{1}{\prod (q)_{M_i - \Delta M_i}} = \delta_{M_1,0} \sum_{m \geq 0} (-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m (q)_{M_2-2m} (q)_{M_3-m}},$$

where

$$(6.8) \quad \Delta M_1 = m_1 + m_{12}, \quad \Delta M_2 = 2m_2 + 2m_{12}, \quad \Delta M_3 = m_2 + m_{12},$$

and

$$(6.9) \quad Q = \frac{1}{2} \sum_{i=1,2} m_i(m_i - 1) + m_{12}(m_{12} - 1) + m_1m_2 + (m_1 + m_2)m_{12}.$$

Equation (6.7) can be proved by shifting $m_2 \rightarrow m_2 - m_{12}$, summing over m_{12} by (1.1) and then performing the sum over m_1 by (6.5).

Despite the existence of examples of q -identities for other than quadratic monomial ideals, we believe the ones corresponding to quadratic monomial ideals are ‘the nicests’ and are the ones most relevant for the application in conformal field theory. In a sequel to this paper we discuss q -identities associated to flag varieties [BH]. The corresponding Hilbert series correspond to the partition functions of quasi-particles in WZW conformal field theories and are the building blocks for characters of affine Lie algebras. In fact, we will argue that the partial Hilbert series of an affinized flag variety is, upto a trivial factor, precisely the modified Hall-Littlewood polynomial.

Flag varieties are defined by an ideal of (non-monomial) quadratic relations. Nevertheless, we will show that, at least as far as the computation of the Hilbert series is concerned, the computations can be reduced to those for quadratic monomial ideals discussed in this paper.

ACKNOWLEDGEMENTS

I would like to thank Omar Foda, Emily Hackett-Jones and David Ridout for discussions. P.B. is supported by a QEII research fellowship from the Australian Research Council.

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DEPARTMENT OF PHYSICS AND MATHEMATICAL PHYSICS, UNIVERSITY OF ADELAIDE, ADELAIDE SA 5005, AUSTRALIA

E-mail address: `pbouwkne@physics.adelaide.edu.au`